Linear Algebra and its Applications 478 (2015) 131–161

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Convergence analysis for Kaczmarz-type methods in a Hilbert space framework

Peter Oswald *, Weiqi Zhou

Jacobs University Bremen, Research I, Campus Ring 1, D-28759 Bremen, Germany

ARTICLE INFO

Article history:
Received 29 October 2014
Accepted 23 March 2015
Available online xxxx
Submitted by D.B. Szyld

MSC:
65F10
65F08
65N22
65H10

Keywords:
Subspace correction
Alternating directions method
POCS algorithms
Gauss–Seidel iteration
Randomized
Condition number
Least-squares problems

ABSTRACT

Using the concept of stable Hilbert space splittings, we provide a unified approach to the convergence analysis for multiplicative Schwarz methods (a version of alternating directions methods), and in particular Kaczmarz-type methods for solving linear systems. We consider both fixed cyclic and randomized ordering strategies, and cover block versions as well. For the classical Kaczmarz method with cyclic ordering for solving general linear systems $Ax = b$, a new convergence rate estimate in terms of the generalized condition number of A and logarithmically depending on the rank of A is presented.

* Corresponding author.
E-mail address: agp.oswald@gmail.com (P. Oswald).

http://dx.doi.org/10.1016/j.laa.2015.03.028
0024-3795/

1. Introduction

This paper is a reaction to a number of recent publications [1–5] on randomized versions of the Kaczmarz method triggered by Strohmer and Vershynin [6], and should be viewed as an addendum to [7,8]. The latter two papers are devoted to the theory of so-called Schwarz iterative (or subspace correction) methods for solving elliptic variational problems in Hilbert spaces. That the Kaczmarz method is a particular instance of Schwarz iterative methods has been pointed out in [8]. Alternatively, the Kaczmarz method is a special case of the Neumann–Halperin alternating directions method (ADM) for finding a point in the intersection of many (affine) subspaces of a Hilbert space [9,10] which in turn is part of the family of projection onto convex sets (POCS) algorithms that is popular in many applications (e.g. [11,12]).

The classical Kaczmarz method (with relaxation parameter) for solving general linear systems $Ax = b$ with given right-hand side $b \in \mathbb{C}^n$ and matrix $A \in \mathbb{C}^{m\times n}$, originally proposed by S. Kaczmarz [13] in 1937 for the case $m = n$, is defined as the iteration

$$x^{j+1} = x^j + \omega_j \frac{b_{i_j} - \underline{a}_{i_j} x^j}{\|\underline{a}_{i_j}\|_2^2} \underline{a}_{i_j}^H, \qquad j = 0, 1, \ldots, \tag{1}$$

where $\underline{a}_i \in \mathbb{C}^n$, $i = 1, \ldots, m$, denote the row vectors of A (thus, $\underline{a}_i^H$ are the column vectors of A^H, the Hermitian conjugate of A), and x^0 is a given starting vector. From the update formula (1) it follows that, if we choose as Hilbert space the subspace $V := \mathrm{Ran}(A^H)$ of $\mathbb{C}^n$, the Kaczmarz iteration can be interpreted as ADM with the m coordinate hyperplanes $M_i := \{x \in \mathbb{C}^n : \ \underline{a}_i x = b_i\}$ to project on. The sequence $\mathcal{I} := \{i_j\}_{j\geq 0}$ determines the ordering in which ortho-projections onto the hyperplanes M_i are carried out. Typical orderings are

- *cyclic*, where the index set $i = 1, \ldots, m$ is repeatedly traversed in a fixed order, e.g., $i_j = j \pmod m + 1$,
- *random*, where i_j is randomly and independently determined according to a fixed probability distribution $\{p_i\}$, or
- *greedy*, where i_j is picked according to residual information, e.g., to maximize $r_i^j := b_i - \underline{a}_i x^j$ in absolute value,

with many further variations possible. An appealing property of the Kaczmarz method is that, under mild conditions on the ordering $\mathcal{I}$ and the relaxation parameters $\omega_j \in (0, 2)$, the iteration (1) is convergent. Moreover, if b belongs to the range of A (consistent case) and $x^0 \in \mathrm{Ran}(A^H)$ then it ultimately converges to the least-squares solution $x_{LS} = A^\dagger b$ of the system $Ax = b$. Here $A^\dagger$ denotes the pseudo-inverse of A.

It is well-known that (1) is equivalent to the successive over-relaxation (SOR) iteration with index sequence $\mathcal{I}$ for the system $AA^H y = b$ if the starting vector $x^0 = A^H y^0$ belongs to $\mathrm{Ran}(A^H)$. Indeed, the j-th SOR step with relaxation parameter ω_j for this system can be written in the form

$$x^j = A^H y^j, \qquad y_i^{j+1} = y_i^j + \begin{cases} \omega_j \frac{b_{i_j} - \underline{a}_{i_j} x^j}{\|\underline{a}_{i_j}\|_2^2}, & i = i_j, \\ 0, & i \neq i_j, \end{cases} \qquad j = 0, 1, \ldots.$$

It is easy to check that then

$$x^{j+1} = A^H y^j + \omega_j \frac{b_{i_j} - \underline{a}_{i_j} x^j}{\|\underline{a}_{i_j}\|_2^2} a_{i_j}^H = x^j + \omega_j \frac{b_{i_j} - \underline{a}_{i_j} x^j}{\|\underline{a}_{i_j}\|_2^2} a_{i_j}^H.$$

The convergence theory of the Kaczmarz method and its block versions is typically approached either via the SOR interpretation, or the previously discussed ADM formulation, even though this is not always made explicit.

Schwarz iterative methods, see [14–16,7] for their origins and an outline of their theory, are essentially a reformulation of ADM within a more constructive framework which was motivated by the need for solving large-scale discretizations of operator equations in Hilbert spaces, such as elliptic partial differential and integral equations in Sobolev spaces. This framework is briefly introduced in Section 2. It leads to generic upper bounds for the convergence speed of Schwarz iterative methods (and thus ADM and, in particular, Kaczmarz methods) for deterministic cyclic [7], greedy, and random orderings [8] in terms of the spectral properties of a transformed operator equation generated by the original problem and its splitting into subproblems. Since the convergence estimate for cyclic orderings obtained in [7] was not proved in full generality, and does not seem to appear in the ADM and POCS literature, we state it here as Theorem 1, and give a short proof of it. We also quote and generalize the convergence estimate for random orderings originated from [6] and extended in [17,8], see Theorem 2. Theorem 3 is new, it concerns a different randomized block version of the Kaczmarz iteration, and provides a link between the randomized Kaczmarz iteration and an iteration with simultaneous (or synchronous) update rule. In Section 3 we show how the convergence rate estimates of many recent papers on the Kaczmarz method and its block versions can be obtained from the existing theory of Schwarz iterative methods in a unified way. We also provide an improved convergence bound for the Kaczmarz method with cyclic orderings, see Theorem 4. Finally, the concluding Section 4 contains some numerical experiments that illustrate and complement the theoretical part.

2. Convergence of Schwarz iterative methods

2.1. Space splittings

We repeat the setup from [8], with the only difference that we explicitly allow all Hilbert spaces to be over the field $\mathbb{C}$ (there is no difference but a notational one to the case of spaces over $\mathbb{R}$ considered in most of the previous papers). Consider a separable Hilbert space V, let $a(\cdot,\cdot)$ be a continuous positive definite Hermitian form on V, and let F be a bounded linear functional on V. We use the notation V_a if we consider V as

Hilbert space with the scalar product given by the form $a(\cdot,\cdot)$. Obviously, knowing the norm $\|v\|_a := \sqrt{a(v,v)}$ in V_a determines $a(v,w)$ for all $v, w \in V_a$. To solve the variational problem, *find* $u \in V$ *such that*

$$a(u,v) = F(v) \qquad \forall v \in V, \tag{2}$$

we use the concept of stable space splittings [16]. Let V_a be represented by an at most countable number of Hilbert spaces V_{a_i} equipped with positive definite Hermitian forms $a_i(\cdot,\cdot)$, and associated bounded linear operators $R_i : V_{a_i} \to V_a$ as follows:

$$V_a = \sum_i R_i V_{a_i} := \{v = \sum_i R_i v_i : \ v_i \in V_{a_i}\}. \tag{3}$$

We allow for redundancy, i.e., we do not assume that V_a is the direct sum of its subspaces $R_i V_{a_i}$. We call (3) a *stable space splitting*, if

$$0 < \lambda_{\min} := \inf_{u \in V_a} \frac{a(u,u)}{|||u|||^2} \ \le \ \lambda_{\max} := \sup_{u \in V_a} \frac{a(u,u)}{|||u|||^2} < \infty, \tag{4}$$

where

$$|||u|||^2 := \inf_{v_i \in V_{a_i}:\ u = \sum_i R_i v_i} \sum_i a_i(v_i, v_i).$$

The constants $\lambda_{\min}$ and $\lambda_{\max}$ are called *lower* and *upper stability constants* respectively, and $\kappa := \lambda_{\max}/\lambda_{\min}$ is called the *condition number* of the space splitting (3).

For better orientation of the reader, we give examples of space splittings related to the solution of linear systems $Ax = b$. The first one addresses the case of positive definite Hermitian matrices, while the remaining two are underlying the treatment of Kaczmarz-type methods for general linear systems we focus on in this paper.

- **Example 1.** The standard space splitting for solving linear systems $Ax = b$ with positive definite Hermitian $A \in \mathbb{C}^{n\times n}$ is given by $V = V_a = \mathbb{C}^n$, with the form $a(x,y) = y^H A x$ induced by A, and

  $$V_{a_i} = \mathbb{C}, \quad \|x_i\|_{a_i}^2 = a_i(x_i, x_i) = a_{ii}|x_i|^2, \qquad R_i x_i = x_i e_i, \qquad i = 1, \dots, n,$$

  where e_i denotes the i-th unit coordinate basis vector in $\mathbb{C}^n$, and a_{ii} the diagonal elements of A. Then

  $$|||x|||^2 = \sum_{i=1}^n a_{ii}|x_i|^2, \qquad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{C}^n,$$

  and the condition of the splitting κ equals the spectral condition number of $D^{-1/2} A D^{-1/2}$, where $D = \mathrm{diag}(A)$. As will be outlined below, this splitting leads

to the classical Jacobi–Richardson and Gauss–Seidel-type. Replacing the diagonal entries a_{ii} by arbitrary constants $d_i > 0$ in the definition of the above space splitting leads to the study of the influence of diagonal scaling, this modification will appear for Kaczmarz methods below.

- **Example 2.** Let now $A \in \mathbb{C}^{m\times n}$ be arbitrary. It is convenient to first consider the special case of a consistent system $Ax = b$ with $b \in \mathrm{Ran}(A)$ (the general case will be discussed in Section 3). Such a system has the general solution $x = x_{LS} + \hat{x}$, where $x_{LS} \in \mathrm{Ran}(A^H)$ is the least-squares solution of $Ax = b$, and $\hat{x} \in \mathrm{Ker}(A)$ is arbitrary, and it is known that the classical Kaczmarz method will converge to x_{LS} if $x^0 \in \mathrm{Ran}(A^H)$. Set $V = V_a := \mathrm{Ran}(A^H) \subset \mathbb{C}^n$ with $a(x,x) = x^H x$. The variational problem (2) is a trivial one: $x = x_{LS}$ (since $x_{LS} \in V_a$ this makes sense). For the auxiliary spaces V_{a_i} and the operators R_i, we set

$$V_{a_i} = \mathbb{C}, \quad \|y_i\|_{a_i}^2 = a_i(y_i, y_i) := d_i|y_i|^2, \qquad R_i y_i = y_i \underline{a}_i^H, \qquad i = 1, \dots, m,$$

where $d_i > 0$ are so far unspecified constants. Let D denote the $m \times m$ diagonal matrix formed by these d_i. A straightforward computation leads to

$$\begin{aligned} |||x|||^2 &= \inf_{y_i:\, x=\sum_{i=1}^m R_i y_i} \sum_{i=1}^m d_i y_i^2 = \inf_{y\in\mathbb{C}^m:\, x=A^H y} \|D^{1/2} y\|_2^2 \\ &= \inf_{z\in\mathbb{C}^m:\, x=A^H D^{-1/2} z} \|z\|_2^2, \qquad x \in V_a. \end{aligned}$$

Thus, the stability constants of this splitting are $\lambda_{\min} = \sigma_{\min}^2(D^{-1/2}A)$ (the smallest nonzero eigenvalue of $A^H D^{-1} A$) and $\lambda_{\max} = \sigma_{\max}^2(D^{-1/2}A)$ (the largest eigenvalue of $A^H D^{-1} A$). Consequently, the condition number of the splitting equals the essential condition number of $A^H D^{-1} A$, i.e.,

$$\kappa = \bar{\kappa}(A^H D^{-1} A) := \sigma_{\max}^2(D^{-1/2}A)/\sigma_{\min}^2(D^{-1/2}A). \tag{5}$$

- **Example 3.** The analysis of block-iterative methods requires different splittings, based either on row or on column partitionings of A. We mention one such splitting that is related to the results about block-Kaczmarz solvers in [4,5]. Consider the same V_a and variational problem (2) as in Example 2. Let $\mathcal{T} := \{\tau_k\}_{k=1}^K$ be a finite partition of the row index set $\{1, 2, \dots, m\}$, and denote the associated $|\tau_k| \times n$ submatrices of A by A_{τ_k}. Define auxiliary positive-definite Hermitian forms on $V_{a_k} = \mathrm{Ran}(A_{\tau_k}) \subset \mathbb{C}^{|\tau_k|}$ by setting

$$a_k(y_{\tau_k}, y_{\tau_k}) = \|A_{\tau_k}^H y_{\tau_k}\|_2^2, \quad y_{\tau_k} \in V_{a_k}, \qquad k = 1, \dots, K.$$

Finally, let $R_k y_{\tau_k} = A^H_{\tau_k} y_{\tau_k}$ be the extension operators from V_{a_k} to V_a, $k = 1, \ldots, K$. Since $\mathrm{Ran}(A^H_{\tau_k} A_{\tau_k}) = \mathrm{Ran}(A^H_{\tau_k})$, by these definitions we have

$$|||x|||^2 = \inf_{y_{\tau_k} \in V_{a_k}:\, x = \sum_{k=1}^K R_k y_{\tau_k}} \|R_k y_{\tau_k}\|_2^2 = \inf_{z_k \in \mathrm{Ran}(A^H_{\tau_k}):\, x = \sum_{k=1}^K z_k} \sum_{k=1}^K \|z_k\|_2^2.$$

For $x \in V_a$ and any decomposition $x = \sum_{k=1}^K z_k$ with $z_k \in \mathrm{Ran}(A^H_{\tau_k})$ we have

$$a(x,x) = \|x\|_2^2 \le (\sum_{k=1}^K \|z_k\|_2)^2 \le K \sum_{k=1}^K \|z_k\|_2^2,$$

which yields the trivial upper bound $\lambda_{\max} = \lambda_{\max,\mathcal{T}} \le K$. However, sharp estimates for the stability constants and condition number $\kappa = \kappa_{\mathcal{T}}$ of this splitting valid for general row partitions are difficult to obtain, even though one would expect a tendency towards improving the condition number $\kappa_{\mathcal{T}}$ when increasing the size of the τ_k. In particular, if $K = 1$ and $\tau_1 = \{1, 2, \ldots, m\}$ then obviously $\kappa_{\mathcal{T}} = 1$.
One case, where the estimation of these constants is relatively easy but does not lead to a small $\kappa_{\mathcal{T}}$, is worth mentioning. Referring to results concerning the optimal paving of operators on Hilbert spaces, the authors of [4,5] consider special row partitions, characterized by the property that there exist positive constants $0 < \alpha < \beta < \infty$ such that

$$\alpha \|y_{\tau_k}\|_2^2 \le \|A^H_{\tau_k} y_{\tau_k}\|_2^2 \le \beta \|y_{\tau_k}\|_2^2, \qquad k = 1, \ldots, K. \tag{6}$$

The existence of such row partitions with $\alpha, \beta \approx 1$ and relatively small K, at least if the rows of A have unit norm, is related to the Bourgain–Tzafriri conjecture (see [18] for a discussion of this and many other conjectures, equivalent to it) which was recently confirmed in [19]. Substituting (6) into the formula for $|||x|||^2$, we conclude that

$$\frac{\|x\|_2^2}{\beta \|(A^H)^\dagger x\|_2^2} = \frac{\|x\|_2^2}{\beta \inf_{y:\, A^H y = x} \|y\|_2^2} \le \frac{\|x\|_2^2}{|||x|||^2} \le \frac{\|x\|_2^2}{\alpha \|(A^H)^\dagger x\|_2^2},$$

and thus

$$\frac{\alpha}{\beta} \bar{\kappa}(A^H A) \le \kappa_{\mathcal{T}} \le \frac{\beta}{\alpha} \bar{\kappa}(A^H A). \tag{7}$$

Therefore, row partitions $\mathcal{T}$ satisfying (6) with β/α close to 1, as discussed in [4,5], do not have any preconditioning effect on solving $Ax = b$ but guarantee fast solvability of the subproblems in block-Kaczmarz iterations considered in Section 3.2.

2.2. Schwarz iterative methods

For the setup of *Schwarz iterative methods* (or subspace correction methods) associated with (3) we restrict ourselves to finite splittings ($i = 1, \dots, N$). We define linear operators $T_i : V_a \to V_{a_i}$ via the variational problems

$$a_i(T_i v, v_i) = a(v, R_i v_i) \qquad \forall\, v_i \in V_{a_i}, \tag{8}$$

to be solved for given $v \in V_a$ in the spaces V_{a_i}, $i = 1, \dots, N$. Using these T_i, analogs of the classical Jacobi–Richardson and Gauss–Seidel-SOR iterations, called additive and multiplicative Schwarz methods associated with the stable space splitting (3) can be introduced pretty much along the lines of the standard methods, see [7,8,16,14,20]. The *additive* (or *parallel* or *synchronous*) *Schwarz iteration* is given by

$$u^{\ell+1} = u^\ell + \omega_\ell \sum_{i=1}^N R_i T_i e^\ell, \qquad e^\ell := u - u^\ell, \qquad \ell \geq 0, \tag{9}$$

where a starting point u^0 needs to be provided, and $u \in V_a$ is the solution of (2). Since

$$a_i(T_i e^\ell, v_i) = a(u - u^\ell, R_i v_i) = F(R_i v_i) - a(u^\ell, R_i v_i),$$

the subproblem results $T_i e^\ell$ are computable from available information, and the update direction

$$w^j := P e^\ell, \qquad P := \sum_{i=1}^N R_i T_i, \tag{10}$$

can easily be computed from the subproblem results. If $\omega_\ell = \omega$ is fixed for all $\ell \geq 0$ then we get the Richardson method for the operator equation

$$Pu = \sum_{i=1}^N R_i f_i, \tag{11}$$

where f_i is defined by the variational problems $a_i(f_i, v_i) = F(R_i v_i)$ valid for all $v_i \in V_{a_i}$ and $i = 1, \dots, N$. If the splitting (3) is stable, then the operator P, called *additive Schwarz operator*, is Hermitian and positive-definite on V_a, and satisfies the identity

$$a(Pv, v) = |||Pv|||^2 = \sum_{i=1}^N a_i(T_i v, T_i v) \qquad \forall\, v \in V_a. \tag{12}$$

From the definition (4) and (12) we see that stability constants and condition number of the splitting (3) are closely related to the spectral properties of P:

$$\lambda_{\min} = \inf_{\|v\|_a=1} a(Pv,v), \qquad \lambda_{\max} = \sup_{\|v\|_a=1} a(Pv,v). \tag{13}$$

Moreover, (11) is equivalent to (2). Thus, the additive Schwarz method converges for $0 < \omega < 2/\lambda_{\max}$, and if $\omega = 2/(\lambda_{\max} + \lambda_{\min})$ we have the estimate for the asymptotically optimal error reduction

$$\|u - u^\ell\|_a \le \|I - \omega P\|^\ell_{V_a \to V_a} \|u - u^0\|_a = \left(1 - \frac{2}{1+\kappa}\right)^\ell \|u - u^0\|_a. \tag{14}$$

The *multiplicative* (or *sequential* or *asynchronous*) *Schwarz iteration* which we focus on in this paper assumes a certain index ordering $\mathcal{I} = \{i_j\}_{j\ge 0}$ and processes subproblems in this order: Given u^0, we recursively determine

$$u^{j+1} = u^j + \omega_j R_{i_j} T_{i_j} e^j, \qquad e^j := u - u^j, \quad j \ge 0. \tag{15}$$

For the space splittings of Example 1, this iteration (15) reduces to SOR type methods. Indeed, denoting by x^j the j-th iterate and by x the solution of $Ax = b$, from (8) we get $T_i e^j = (a_{ii})^{-1}(Ax - Ax^j)_i = (a_{ii})^{-1}(b_i - \sum_{k=1}^n a_{ik} x^j_k)$, which means that $x^{j+1}_k = x^j_k$ for $k \ne i_j$, and

$$x^{j+1}_{i_j} = x^j_{i_j} + \frac{\omega_j}{a_{i_j i_j}}(b_{i_j} - \sum_{k=1}^n a_{i_j k} x^j_k).$$

This is the SOR update for the i_j-th equation, and in particular the Gauss–Seidel update if $\omega_j = 1$. The splittings from Examples 2 and 3 cover the classical and block versions of the Kaczmarz iteration, this will be discussed in Sections 3.1 and 3.2 respectively.

For a fair comparison with (9), one usually lumps N steps of the recursion (15) together into one sweep, and compares their joint error reduction effect with the error reduction (9) in one step of the additive Schwarz iteration. For cyclic orderings it is often observed in practice that in this comparison the multiplicative Schwarz iteration is superior to the additive Schwarz iteration. However, this has been substantiated only for special problem classes, and is, in general, not true, see [16]. Finding sharp estimates for the convergence of the iteration (15) with cyclic ordering is, despite many attempts [21,7,20], not yet in a final state. The convergence theory for (15) has drawn renewed attention after Strohmer and Vershynin [6] proved, in an elementary way, a general and realistic bound for the error decay in expectation of a randomized version of the Kaczmarz method. This result was immediately taken up and extended in various directions, see, e.g., [1,17,8]. We also note that related developments happened independently in the convex optimization community, e.g., for direct search methods, see [22,23]. A similarly elementary convergence rate estimate holds for the iteration (15) with greedy orderings (see [8], to not overload the present paper, we will not dwell on greedy versions).

In the remainder of this section, we will state general estimates for the relative error reduction in multiplicative Schwarz iterations. The first one, for the standard cyclic ordering with constant relaxation $\omega_j = \omega$ is essentially contained in [7], where it is proved for the special case of subspaces $V_i \subset V$, and R_i being the natural injections. To make the paper self-contained, we include the proof. The second result is based on results in [6,17] and quoted from [8]. It concerns randomized orderings, i.e., in each step we choose an $i_j \in \{1, \ldots, N\}$ according to a fixed discrete probability distribution. We also state a convergence estimate for a slight extension of the algorithm (15), where in the j-th step an index group $I_j \subset \{1, \ldots, N\}$ of size $1 \le k_j \le N$ is picked, and an update similar to the one in (9) using subproblem solutions for all $i \in I_j$ is performed. Such a modification has been mentioned without proof in [8] and might prove useful in further optimizing the performance of randomized algorithms for large-scale linear systems. It provides a link between the randomized multiplicative Schwarz iteration ($k_j = 1$) and the additive Schwarz iteration (9) which corresponds to the case $k_j = N$.

2.3. Cyclic orderings

In this subsection, we consider cyclic orderings $\mathcal{I}$ given by $i_j = j \pmod N + 1$, $j \ge 0$. The relaxation parameters are constant: $\omega_j = \omega \in (0, 2)$. Before stating the result, we make some theoretical assumptions on norm estimates for the operators $R_i : V_{a_i} \to V_a$. In particular, assume we know positive constants $\gamma_i \ge \|R_i\|^2_{V_{a_i} \to V_a}$ such that

$$a(R_i v_i, R_i v_i) \le \gamma_i a_i(v_i, v_i) \qquad \forall\, v_i \in V_{a_i}, \quad i = 1, \ldots, N. \tag{16}$$

Also, let $\gamma \ge \lambda_{\max}$ be a given upper bound for the upper stability constant of the splitting (3). We then have

$$a(\sum_{i \in I} R_i v_i, \sum_{i \in I} R_i v_i) \le \gamma \sum_{i \in I} a_i(v_i, v_i) \tag{17}$$

for all index subsets $I \subset \{1, \ldots, N\}$ and all $v_i \in V_{a_i}$ (just set $u = \sum_{i \in I} R_i v_i$, and look at the definition of $|||u|||^2$ after (4)). In some cases (as for the Kaczmarz method, see Section 3.1), such constants can be computed explicitly, in others knowledge about them for the execution of the algorithm can be circumvented at little extra cost (e.g., by switching to steepest descent updates). To satisfy (16) and (17), in theory we can always take

$$\gamma_i = \gamma = \lambda_{\max}. \tag{18}$$

The following theorem has been stated in [7] for the situation when $V_{a_i} \subset V$, and the mappings R_i are the natural injections. We repeat it for the present setting, also because it seems not widely known.

Theorem 1. *Assume that (3) is a stable space splitting of the Hilbert space V_a, with stability constants $\lambda_{\min\,/\,\max}$, and condition number κ given by (4). Then:*

a) *The multiplicative Schwarz iteration (15) with standard cyclic ordering $\mathcal{I}$ be given by $i_j = j \pmod N + 1$, $j \ge 0$, and constant relaxation parameters $\omega_j = \omega \in (0, 2/\lambda_{\max})$ converges to the solution u of (2), with error decay given by*

$$\|u - \bar{u}^\ell\|_a^2 \le \left(1 - \frac{C_0}{\kappa}\right)^\ell \|u - u^0\|_a^2, \tag{19}$$

where $\bar{u}^\ell = u^{\ell N}$ is the solution after ℓ sweeps and

$$C_0 := \frac{\omega\lambda_{\max}(2 - \omega\lambda_{\max})}{(\frac{1}{2}\lfloor \log_2(2N)\rfloor \omega\lambda_{\max} + 1)^2}.$$

b) *Depending on N and $\lambda_{\max}$, the relaxation parameter ω can be chosen such that*

$$\|u - \bar{u}^\ell\|_a^2 \le \left(1 - \frac{1}{(\lfloor \log_2(2N)\rfloor + 1)\kappa}\right)^\ell \|u - u^0\|_a^2. \tag{20}$$

Proof. Let $\tilde{V}_{\tilde{a}} := V_{a_1} \oplus V_{a_2} \oplus \ldots \oplus V_{a_N}$ be the direct sum of the Hilbert spaces $\{V_{a_i}\}$, with elements denoted by $\tilde{v} = (v_1, v_2, \ldots, v_N)$, and scalar product given by

$$\|\tilde{v}\|_{\tilde{a}} = \tilde{a}(\tilde{v}, \tilde{v}) := \sum_{i=1}^N a_i(v_i, v_i), \qquad v_i \in V_{a_i}, \quad i = 1, 2, \ldots, N.$$

Define the linear operator $R : \tilde{V}_{\tilde{a}} \longmapsto V_a$ by the formula

$$R\tilde{v} = \sum_{i=1}^N R_i v_i, \qquad \tilde{v} \in \tilde{V}_{\tilde{a}}.$$

According to (8), its adjoint $R^* : V_a \longmapsto \tilde{V}_{\tilde{a}}$ can be expressed by the operators T_i:

$$R^* v = (T_1 v, T_2 v, \ldots, T_N v), \qquad v \in V_a.$$

Introduce the linear operator

$$\tilde{P} = R^* R = \begin{pmatrix} T_1R_1 & T_1R_2 & \ldots & T_1R_N \\ T_2R_1 & T_2R_2 & \ldots & T_2R_N \\ & \cdot & & \\ \vdots & \vdots & \ddots & \vdots \\ T_NR_1 & T_NR_2 & \ldots & T_NR_N \end{pmatrix},$$

which acts boundedly on $\tilde{V}_{\tilde{a}}$, and is the counterpart of the additive Schwarz operator $P = RR^*$ defined in (10). By the spectral properties of P, see (12) and (13), we get the lower estimate

$$\tilde{a}(R^*R\tilde{v}, R^*R\tilde{v}) = a(PR\tilde{v}, R\tilde{v}) \geq \lambda_{\min} a(R\tilde{v}, R\tilde{v}) = \lambda_{\min} \|R\tilde{v}\|_a^2, \tag{21}$$

and similarly the norm bound

$$\|\tilde{P}\|_{\tilde{V}_{\tilde{a}} \to \tilde{V}_{\tilde{a}}} \leq \lambda_{\max}, \tag{22}$$

since

$$\|\tilde{P}\tilde{v}\|_{\tilde{a}}^2 = a(PR\tilde{v}, R\tilde{v}) \leq \lambda_{\max} a(R\tilde{v}, R\tilde{v}) = \lambda_{\max} \tilde{a}(\tilde{P}\tilde{v}, \tilde{v}) \leq \lambda_{\max} \|\tilde{P}\tilde{v}\|_{\tilde{a}} \|\tilde{v}\|_{\tilde{a}}.$$

We can decompose $\tilde{P}$ into strictly lower triangular, diagonal, and strictly upper triangular parts

$$\tilde{P} = \tilde{D} + \tilde{L} + \tilde{L}^*,$$

where

$$\tilde{L} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ T_2R_1 & 0 & 0 & \dots & 0 \\ T_3R_1 & T_3R_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ T_NR_1 & T_NR_2 & T_NR_3 & \dots & 0 \end{pmatrix},$$

$$\tilde{D} = \begin{pmatrix} T_1R_1 & 0 & 0 & \dots & 0 \\ 0 & T_2R_2 & 0 & \dots & 0 \\ 0 & 0 & T_3R_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & T_NR_N \end{pmatrix}.$$

Since

$$\begin{aligned} \|\tilde{D}\tilde{v}\|_{\tilde{a}}^2 &= \sum_{i=1}^N a_i(T_iR_iv_i, T_iR_iv_i) = \sum_{i=1}^N a(PR_iv_i, R_iv_i) \\ &\leq \lambda_{\max} \sum_{i=1}^N a(R_iv_i, R_iv_i) = \lambda_{\max} \sum_{i=1}^N a_i(T_iR_iv_i, v_i) \leq \lambda_{\max} \|\tilde{D}\tilde{v}\|_{\tilde{a}} \|\tilde{v}\|_{\tilde{a}}, \end{aligned}$$

we have

$$\|\tilde{D}\|_{\tilde{V}_{\tilde{a}} \to \tilde{V}_{\tilde{a}}} \leq \lambda_{\max}. \tag{23}$$

Moreover, as established in [7, Theorem 4], we have the following estimate for the lower triangular operator $\tilde{L}$:

$$\|\tilde{L}\|_{\tilde{V}_{\tilde{a}}\to\tilde{V}_{\tilde{a}}} \le \frac{1}{2}\lfloor \log_2(2N)\rfloor \|\tilde{P}\|_{\tilde{V}_{\tilde{a}}\to\tilde{V}_{\tilde{a}}}, \tag{24}$$

which combined with (22) implies

$$\|\tilde{L}\|_{\tilde{V}_{\tilde{a}}\to\tilde{V}_{\tilde{a}}} \le \frac{1}{2}\lfloor \log_2(2N)\rfloor \lambda_{\max}. \tag{25}$$

We note that counterparts of (24) for the matrix case have been investigated a lot, and that the logarithmic dependence of the bound on N cannot be improved asymptotically in some instances.

With this notation at hand, we can reformulate (2) as variational problem on $\tilde{V}_{\tilde{a}}$, namely, find $\tilde{u} \in \tilde{V}_{\tilde{a}}$ such that

$$\tilde{a}(\tilde{P}\tilde{u}, \tilde{v}) = a(R\tilde{u}, R\tilde{v}) = F(R\tilde{v}) \qquad \forall\ \tilde{v} \in \tilde{V}_{\tilde{a}},$$

and view the cyclic Schwarz iteration in $\tilde{V}_{\tilde{a}}$ as an SOR-type iteration on a linear equation of the form $\tilde{P}\tilde{u} = \tilde{b}$, where the right-hand side $\tilde{b}$ satisfies $\tilde{a}(\tilde{b}, \tilde{v}) = F(R\tilde{v})$ for all $\tilde{v} \in \tilde{V}_{\tilde{a}}$. Details can be found in [16,7]. The result is the following formula for the error propagation per sweep of (15) with cyclic ordering, where as before we denote by $\bar{u}^\ell = u^{\ell N}$ the iterate after the ℓ-th sweep, and by u the solution of (2):

$$\bar{u}^{\ell+1} - u = (I - \omega R(\tilde{I} + \omega\tilde{L})^{-1}R^*)(\bar{u}^\ell - u), \qquad \ell \ge 0.$$

Here I and $\tilde{I}$ denote the identity operators on V_a and $\tilde{V}_{\tilde{a}}$ respectively. Thus, the error decay per sweep of the cyclic Schwarz iteration is determined by the error propagation operator

$$Q = I - \omega R(\tilde{I} + \omega\tilde{L})^{-1}R^* :\ V_a \to V_a. \tag{26}$$

To estimate its norm as map in V_a, we use the identity

$$\begin{aligned} Q^*Q &= (I - \omega R(\tilde{I} + \omega\tilde{L})^{-*}R^*)(I - \omega R(\tilde{I} + \omega\tilde{L})^{-1}R^*) \\ &= I - \omega R(\tilde{I} + \omega\tilde{L})^{-*}(\tilde{I} + \omega\tilde{L} + \tilde{I} + \omega\tilde{L}^* - \omega R^*R)(I + \omega L)^{-1}R^* \\ &= I - \omega R(\tilde{I} + \omega\tilde{L})^{-*}(2\tilde{I} - \omega\tilde{D})(\tilde{I} + \omega\tilde{L})^{-1}R^*. \end{aligned}$$

Thus,

$$\begin{aligned} \|Q\|^2_{V_a\to V_a} &= \sup_{\|v\|_a=1} a(Q^*Qv, v) \\ &= 1 - \omega \inf_{\|v\|_a=1} \tilde{a}((2\tilde{I} - \omega\tilde{D})(\tilde{I} + \omega\tilde{L})^{-1}R^*v, (\tilde{I} + \omega\tilde{L})^{-1}R^*v) \end{aligned}$$

$$\leq 1 - \omega(2 - \omega\lambda_{\max}) \inf_{\|v\|_a=1} \tilde{a}((\tilde{I} + \omega\tilde{L})^{-1}R^*v, (\tilde{I} + \omega\tilde{L})^{-1}R^*v)$$

$$\leq 1 - \frac{\omega(2 - \omega\lambda_{\max})}{(\frac{1}{2}\omega\lfloor\log_2(2N)\rfloor\lambda_{\max} + 1)^2} \inf_{\|v\|_a=1} \tilde{a}(R^*v, R^*v)$$

$$\leq 1 - \frac{\omega(2 - \omega\lambda_{\max})\lambda_{\min}}{(\frac{1}{2}\omega\lfloor\log_2(2N)\rfloor\lambda_{\max} + 1)^2}.$$

The first inequality follows from (23), the second from (25), and the last from (21). This proves (19).

Straightforward minimization leads to (20) if we choose ω according to

$$\omega = \frac{1}{(\frac{1}{2}\lfloor\log_2(2N)\rfloor + 1)\lambda_{\max}}. \tag{27}$$

This establishes Theorem 1. □

Remarks. 1) The estimates (19) and (20) for the convergence rate of the multiplicative Schwarz iteration are uniform with respect to ordering – all quantities entering it do not change if the enumeration of subproblems is changed. This is a drawback of the estimation technique. Numerical examples show that the convergence rate of the cyclic Schwarz iteration may significantly change when the subproblems are randomly reordered before execution (this may serve as another indication for the sometimes observed speed-up when random orderings $\mathcal{I}$ are used).

2) An advantage of our bounds (19) and (20) is that they highlight the dependence of convergence rates on spectral properties of the operator P, and in particular on its spectral condition number, which equals the condition number κ of the underlying space splitting (3). This makes them also comparable with the recently obtained bounds for similar iterations using randomized and greedy orderings. Previous convergence estimates for cyclic orderings, most notably the Whitney–Meany estimate [10, Theorem 2.77, Theorem 4.4] and the results in [20], both based on analyzing the product representation

$$Q = (I - \omega R_N T_N) \dots (I - \omega R_1 T_1)$$

of the error propagation operator (26), are different in nature and, in many cases, weaker than (20).

2.4. Random orderings

The following theorem can be found in [8, Theorem 1, b)], it generalizes the results of [6,17] to the case of Schwarz iterations based on space splittings. Its proof is completely elementary, and will not be repeated here (see the proof of Theorem 3 below for a similar argument).

Theorem 2. *Assume that (3) is a stable space splitting of the Hilbert space V_a, with lower stability constant $\lambda_{\min}$ given by (4), and assume that the constants $\gamma_i > 0$, $i = 1, \ldots, N$, satisfy (16). Create a random ordering $\mathcal{I}$ by setting $i_j = i$ with probability $p_i = \gamma_i/(\gamma_1 + \ldots + \gamma_N)$, independently for different $j \geq 0$. Finally, set $\omega_j = \omega/\gamma_{i_j}$, where $\omega \in (0, 2)$ is fixed.*

Then the multiplicative Schwarz iteration (15) with random ordering $\mathcal{I}$ converges in expectation with the expected error decay given by

$$E(\|u - u^j\|_a^2) \leq \left(1 - \frac{\omega(2-\omega)\lambda_{\min}}{\gamma_1 + \ldots + \gamma_N}\right)^j \|u - u^0\|_a^2, \qquad j \geq 1. \tag{28}$$

At first glance, the estimates suggest that $\omega = 1$ is the best choice for the relaxation parameter, even though it is well-known that for certain applications, over- ($\omega > 1$) or under-relaxation ($\omega < 1$) pays off. The question of choosing ω is intertwined with our choice of the probability distribution p_i which is determined from the γ_i defined by (16). This is related to the problem of optimal scaling of the subproblems in V_{a_i} which does not have a trivial solution in general (see [24,25] for recent discussions of the scaling aspect).

If we choose equal $\gamma_i = \lambda_{\max}$ as in (18), then (28) implies the estimate

$$E(\|u - u^j\|_a^2) \leq (1 - \frac{c_0}{N\kappa})^j \|u - u^0\|_a^2, \qquad j \geq 1,$$

where $c_0 = \omega(2-\omega) \leq 1$. Therefore, N steps of this randomized multiplicative Schwarz iteration correspond to one sweep, and thus comparable to one step of the additive Schwarz iteration (9), the expected square energy error reduction is roughly bounded by a constant factor

$$(1 - 1/(N\kappa))^N \approx e^{-1/\kappa} \approx 1 - 1/\kappa, \tag{29}$$

if $\kappa >> 1$ and $\omega = 1$. This is qualitatively as good as the estimate (14). Note that (28)–(29) represent upper bounds for the expected convergence rate, whereas (14) is asymptotically sharp and deterministic. The estimate (28) is superior to (14) if $\sum_i \gamma_i << N\lambda_{\max}$.

We present next a more general *block-random Schwarz iteration*, the j-th step of which is as follows: Instead of picking a single index $i_j \in \{1, \ldots, N\}$, we now pick (randomly and uniformly, and independently for different j) a whole index set $I_j \subset \{1, \ldots, N\}$ of size $k_j \in \{1, \ldots, N\}$, and update according to

$$u^{j+1} = u^j + \omega_j \sum_{i \in I_j} R_i T_i e^j. \tag{30}$$

The case $k_j = 1$ corresponds to (15) with random ordering, while $k_j = N$ is equivalent to (9).

Theorem 3. *Assume that (3) is a stable space splitting of the Hilbert space V_a, with stability constants $\lambda_{\min\,/\,\max}$ and condition number κ given by (4). Assume $\omega_j = \omega \in (0, 2/\lambda_{\max})$, and let the random index sets I_j of size k_j be generated as described above.*

Then the modified Schwarz iteration (30) converges in expectation, and the expected error decays according to

$$E(\|u - u^{j+1}\|_a^2) \le \left(1 - \frac{C_1 k_j}{N\kappa}\right) E(\|u - u^j\|_a^2), \qquad j \ge 0, \tag{31}$$

where $C_1 = \omega\lambda_{\max}(2 - \omega\lambda_{\max}) \in (0, 1]$.

Proof. For the following calculations, recall that $a(v, R_i v_i) = a_i(T_i v, v_i)$ for all $v \in V_a$ and $v_i \in V_{a_i}$, and that (17) holds for any $\gamma \ge \lambda_{\max}$. For given u^j, ω_j, and a randomly chosen I_j according to (30) we have

$$\begin{aligned}
\|e^{j+1}\|_a^2 &= a(e^j - \omega \sum_{i\in I_j} R_i T_i e^j, e^j - \omega \sum_{i\in I_j} R_i T_i e^j) \\
&= \|e^j\|_a^2 - 2\omega \sum_{i\in I_j} a(e^j, R_i T_i e^j) + \omega^2 a(\sum_{i\in I_j} R_i T_i e^j, \sum_{i\in I_j} R_i T_i e^j) \\
&\le \|e^j\|_a^2 - 2\omega \sum_{i\in I_j} a_i(T_i e^j, T_i e^j) + \omega^2 \lambda_{\max} \sum_{i\in I_j} a_i(T_i e^j, T_i e^j) \\
&= \|e^j\|_a^2 \left(1 - \omega(2 - \omega\lambda_{\max}) \frac{\sum_{i\in I_j} a_i(T_i e^j, T_i e^j)}{\|e^j\|_a^2}\right) \\
&= \|e^j\|_a^2 \left(1 - \frac{C_1}{\lambda_{\max}} \frac{\sum_{i\in I_j} a_i(T_i e^j, T_i e^j)}{\|e^j\|_a^2}\right),
\end{aligned}$$

where in the inequality step (17) was used. Now recall that $I_j \subset \{1, \dots, N\}$ is a uniformly chosen random subset of k_j indices. This, and the inequality

$$\lambda_{\min} a(v, v) \le a(Pv, v) = \sum_{i=1}^N a_i(T_i v, T_i v) \qquad \forall\, v \in V_a,$$

implied by the lower spectral bound of P, give the following bound for the conditional expectation of $\|e^{j+1}\|_a^2$, given the current error e^j:

$$\begin{aligned}
E(\|e^{j+1}\|_a^2 \,|\, e^j) &\le \|e^j\|_a^2 \left(1 - \frac{C_1}{\lambda_{\max}} \frac{\frac{k_j}{N} \sum_{i=1}^N a_i(T_i e^j, T_i e^j)}{\|e^j\|_a^2}\right) \\
&\le (1 - \frac{C_1 k_j \lambda_{\min}}{N\lambda_{\max}}) \|e^j\|_a^2 = (1 - \frac{C_1 k_j}{N\kappa}) \|e^j\|_a^2.
\end{aligned}$$

Taking expectations with respect to e^j on both sides, we arrive at (31). Theorem 3 is established. □

Remarks. 3) Taking into account that one update step (30) is essentially equivalent to k_j single steps in (15), the upper bound (31) leads to an expected error reduction per sweep comparable with (29). Indeed, assuming $k_l/(N\kappa) << 1$ the guaranteed relative error reduction factor after j-steps of the block-random Schwarz iteration (30) is given by

$$\prod_{l=1}^{j}(1 - \frac{C_1 k_{l-1}}{N\kappa}) \approx (1 - \frac{C_1(k_0 + \ldots + k_{j-1})}{N\kappa}).$$

The numerical experiments reported in Section 4 are confirming this.

4) As already noted, relaxation can boost convergence. However, there is no general recipe for choosing ω_j optimally. For the considered random iterations, computing ω_j by the steepest decent formula

$$\omega_j = \frac{a(e^j, w^j)}{a(w^j, w^j)}, \qquad e_j = u - u^j, \tag{32}$$

in an update step of the form $u^{j+1} = u^j + \omega_j w^j$ is a provably good alternative. This is because steepest decent guarantees maximal error reduction in the given search direction w^j, and thus any of the above recursive estimates for expected square errors will hold, with best possible constants, for the steepest decent update as well.

5) Although the presented Schwarz iteration framework is essentially equivalent to ADM, it is more constructive by emphasizing the component structure of the iterations, and suggests optimization rules. Indeed, in order to arrive at efficient methods, one needs to have an as small as possible condition number κ of the underlying space splitting and, at the same time, cheap components (execution of T_i, involving residual computations and subproblem solves, and R_i). In many problems, slow convergence is due to bad conditioning of the space splitting underlying the given iterative method, and can only be cured by some kind of preconditioning, e.g., by changing the splitting. For elliptic PDE solvers, this approach has been proven very successful. Another aspect is to realize that the auxiliary spaces V_{a_i} need not form direct sum decompositions nor be even subspaces of V_a, and that the subproblems defined by the auxiliary forms $a_i(\cdot,\cdot)$ may not be directly related to the original problem (2). E.g., in applications to solving linear systems we may easily allow for overlapping block covers rather than block partitions of A, and approximate subproblem solves.

3. Applications to Kaczmarz iterations

In order to apply the Schwarz iteration theory based on a Hilbert space setting for the solution (in a least-square sense) of a linear system $Ax = b$ with arbitrary $A \in \mathbb{C}^{m\times n}$, one usually considers the normal equations $A^H Ax = A^H b$, or the system $AA^H y = b$, each having Hermitian positive semi-definite coefficient matrices. The latter formulation is behind the Kaczmarz-type algorithms considered in this section. Our goal is to demonstrate how various recent results on Kaczmarz-type iterations [6,1,4,5,3] can be obtained

in a unified way from the convergence theory for Schwarz iterative methods outlined in Section 2 by applying it to the space splittings of Examples 2 and 3. In addition, we provide an improved convergence rate estimate for cyclic orderings.

3.1. Kaczmarz methods: single row updates

It is convenient to first consider the special case of a consistent system $Ax = b$ with $b \in \mathrm{Ran}(A)$. Such a system has the general solution $x = x_{LS} + \hat{x}$, where $x_{LS} \in \mathrm{Ran}(A^H)$ is the least-squares solution and $\hat{x} \in \mathrm{Ker}(A)$ is arbitrary. The general case will be considered at the end of this section. Example 2 of Section 2.1 is the space splitting which provides the framework for analyzing Kaczmarz methods with single row updates as multiplicative Schwarz method. Using the notation introduced in Section 2, a straightforward computation leads to $T_i x = d_i^{-1} \underline{a}_i x$ for $x \in V_a$ and all $i = 1, \dots, m$. Thus, $T_i e^j = T_i(x_{LS} - x^j) = d_i^{-1}(b_i - \underline{a}_i x^j)$, and (15) specializes to

$$x^{j+1} = x^j + \omega_j \frac{b_{i_j} - \underline{a}_{i_j} x^j}{d_{i_j}} \underline{a}_{i_j}^H, \tag{33}$$

which is identical with the Kaczmarz iteration (1) if the diagonal scaling is set to $d_i = \|\underline{a}_i\|_2^2$.

Throughout the rest of this section we will silently assume that the starting vector of the iterations satisfies $x^0 \in V_a$. Indeed, for arbitrary x^0, it is obvious from the update formula (33) that

$$x^j = \tilde{x}^j + \hat{x}^0,$$

where $x^0 = \tilde{x}^0 + \hat{x}^0$ is the orthogonal decomposition of the starting vector into $\tilde{x}^0 \in V_a$ and $\hat{x}^0 \in V_a^\perp = \mathrm{Ker}(A)$, and $\tilde{x}^j$ denotes the iterates with starting vector $\tilde{x}^0$. Thus, since under the conditions discussed below the iterates $\tilde{x}^j$ converge to x_{LS}, we have $x^j \to x_{LS} + \hat{x}^0$, and all convergence rate estimates stay in place with $\hat{x}^0$ properly subtracted. Choosing $x^0 = 0$ is a universally safe choice.

Convergence estimates: cyclic orderings. Consider the cyclic ordering $i_j = j \pmod m + 1$, $j \geq 0$, and choose a constant relaxation parameter $\omega_j = \omega$. Under the above assumptions on b and x^0, Theorem 1 implies that the Kaczmarz iterates x^j defined by (33) converge to the least-squares solution x_{LS} of $Ax = b$ for $0 < \omega < 2/\lambda_{\max}$, with an error decay rate given by (19). Moreover, as a consequence of (20) and (5), the error decay after ℓ sweeps is given by

$$\|x_{LS} - \bar{x}^\ell\|_2^2 \leq \left(1 - \frac{1}{(\lfloor \log_2(2m) \rfloor + 1)\bar{\kappa}(A^H D^{-1} A)}\right)^\ell \|x_{LS} - x^0\|_2^2, \qquad \ell \geq 1, \tag{34}$$

if ω is chosen appropriately, e.g., according to (27). We are not aware of any appearance of such an error decay estimate in terms of $\bar{\kappa}(A^H D^{-1} A)$ and logarithmically depending on the dimension m of A in the literature.

One may, however, wonder if the estimate (34) can be improved if $r := \mathrm{rank}(A) = \dim V_a << m$. The answer is yes, and may be interesting in cases when the original system $Ax = b$ is heavily overdetermined.

Theorem 4. *The cyclic Kaczmarz iteration (33) with appropriately chosen relaxation parameter ω and row scaling induced by D for solving a linear system $Ax = b$ with $\mathrm{rank}(A) = r \le \min(n, m)$ possesses an error bound*

$$\|x_{LS} - \bar{x}^\ell\|_2^2 \le \left(1 - \frac{C}{(\ln(r)+1)\bar{\kappa}(A^H D^{-1} A)}\right)^\ell \|x_{LS} - x^0\|_2^2, \qquad \ell \ge 1, \tag{35}$$

where C is an absolute constant.

Proof. Following step by step the proof of Theorem 1 applied to our situation, one easily sees that (35) follows if the inequality (24) is replaced by the estimate

$$\|L_B\|_2 \le C(1 + \ln(r))\|B\|_2, \tag{36}$$

applied to the matrix $B = AD^{-1}A^H$ playing the role of $\tilde{P}$. Here, L_B denotes the strictly lower-triangular part of B, and $\|\cdot\|_2$ stands for the ℓ^2-induced matrix norm. We will show that (36) indeed holds with some absolute constant C for arbitrary positive semi-definite Hermitian matrices B of rank r.

Without loss of generality, assume that $\|B\|_2 = \lambda_1 = 1$, where $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_r > 0$ and $\lambda_k = 0$ for $k = r+1, \ldots, m$ are the eigenvalues of B. The main ingredient is an estimate for the Schatten p-norm $\|L\|_{\sigma,p}$ of strictly lower-triangular matrices L:

$$\|L\|_{\sigma,p} \le C_0 p \|L + L^H\|_{\sigma,p}, \qquad 2 \le p < \infty,$$

where C_0 is an absolute constant, independent of p and m. It follows from Macaev's Theorem proved in a more general setting in, e.g., [26, Theorem 6.2], where also estimates for C_0 are given. Recall that the Schatten p-norm of a matrix is defined as the ℓ^p norm of its singular values. Thus, if D_B denotes the diagonal part of B then

$$\|D_B\|_{\sigma,p} = (\sum_{i=1}^m |b_{ii}|^p)^{1/p} \le (\sum_{i=1}^m b_{ii})^{1/p} = (\sum_{j=1}^r \lambda_j)^{1/p} \le r^{1/p}.$$

Here we have used that $0 \le b_{ii} \le \lambda_1 = 1$. Similarly,

$$\|B\|_{\sigma,p} = (\sum_{j=1}^r \lambda_j^p)^{1/p} \le r^{1/p}.$$

Combining the last three inequalities, we obtain

$$\|L_B\|_2 \le \|L_B\|_{\sigma,p} \le C_0 p \|L_B + L_B^H\|_{\sigma,p} \le C_0 p(\|B\|_{\sigma,p} + \|D_B\|_{\sigma,p}) \le 2C_0 p r^{1/p}.$$

Choosing here $p = \ln(r)$, and taking into account the trivial bound

$$\|L_B\|_2 \le \|L_B\|_F \le \frac{1}{\sqrt{2}}\|B\|_F = \frac{1}{\sqrt{2}}\|B\|_{\sigma,2} \le \sqrt{\frac{r}{2}},$$

establishes (36) for all r, and proves the statement of Theorem 4. □

Convergence estimates: random orderings. We now apply Theorems 2 and 3. For the splitting from Example 2, we have $\gamma_i := d_i^{-1}\|\underline{a}_i\|_2^2$, $i = 1, \ldots, m$. Theorem 2 suggests the selection of the random index sequence $\mathcal{I}$ using the discrete probability distribution

$$p_i = \frac{\gamma_i}{\gamma_1 + \ldots + \gamma_m} = \frac{\|\underline{a}_i\|_2^2}{d_i \|D^{-1/2}A\|_F^2}, \qquad i = 1, \ldots, m, \tag{37}$$

and the update formula

$$x^{j+1} = x^j + \omega \frac{b_{i_j} - \underline{a}_{i_j} x^j}{\|\underline{a}_{i_j}\|_2^2} \underline{a}_{i_j}^H, \tag{38}$$

i.e., again (1) with fixed $\omega_j = \omega \in (0, 2)$. The expected square error estimate then reads

$$E(\|x_{LS} - x^j\|_2^2) \le \left(1 - \frac{\omega(2-\omega)\sigma_{\min}^2(D^{-1/2}A)}{\|D^{-1/2}A\|_F^2}\right)^j \|x_{LS} - x^0\|_2^2, \qquad j \ge 1, \tag{39}$$

which can be upper-bounded for $\omega = 1$ by

$$E(\|x_{LS} - x^j\|_2^2) \le \left(1 - \frac{1}{r\bar{\kappa}(A^H D^{-1} A)}\right)^j \|x_{LS} - x^0\|_2^2, \qquad j \ge 1, \tag{40}$$

where $r = \dim V_a \le \min(n, m)$. For $d_i = 1$, (39) is the result of [6], the introduction of a diagonal scaling has also been discussed there. Recall that for a fair comparison of the error estimates for cyclic versus random orderings, we need to combine m steps of the randomized iteration into one sweep, yielding a reduction factor of the expected square error per sweep of approximately $\approx 1 - cm/(r\kappa)$ in the randomized iteration, compared to a reduction of the square error per sweep of $1 - C/((\ln(r) + 1)\kappa)$ for cyclic ordering, where $\kappa = \bar{\kappa}(A^H D^{-1} A)$.

Finally, for the application of Theorem 3, assume for simplicity constant $\omega_j = \omega$. Then the update formula (30) reads

$$x^{j+1} = x^j + \omega w^j, \qquad w^j := \sum_{i \in I_j} \frac{b_i - \underline{a}_i x^j}{d_i} \underline{a}_i^H, \tag{41}$$

where $I_j \subset \{1,\dots,m\}$ is a uniformly and independently chosen random index set of size k_j, and the error estimate per step (31) for the expected squared error gives

$$E(\|x_{LS}-x^{j+1}\|_2^2) \le \left(1-\frac{k_j}{m\bar\kappa(A^HD^{-1}A)}\right)E(\|x_{LS}-x^j\|_2^2), \qquad j\ge 0, \tag{42}$$

if we set $\omega = 1/\lambda_{\max}$.

As was remarked in Section 2.4, the estimates (39), (40), and (42) remain true if ω_j is determined from the steepest descent rule (32). This makes any preknowledge about $\lambda_{\max}$ superfluous.

Inconsistent systems. The inconsistent (or noisy) case $b \notin \mathrm{Ran}(A)$ is practically important (note that in principal, due to rounding errors, any system $Ax = b$ needs to be considered inconsistent). Let $\hat b$ be the projection of b onto $\mathrm{Ker}(A^H) \perp \mathrm{Ran}(A)$, and set $\tilde b = b - \hat b$. Using the decomposition $b = \hat b + \tilde b$ and the fact that $Ax = \tilde b$ is a consistent system with the same least-squares solution x_{LS} as $Ax = b$, we can easily extend the analysis to the inconsistent case. E.g., according to (33), in the cyclic case we can write

$$x^{j+1} = x^j + \omega\frac{\tilde b_{i_j} - \underline{a}_{i_j}x^j}{d_{i_j}}\underline{a}_{i_j}^H + \frac{\omega\hat b_{i_j}}{d_{i_j}}\underline{a}_{i_j}^H,$$

which gives an inhomogeneous recursion for the error $e^j = x_{LS} - x^j$ of the form

$$e^{j+1} = e^j - \omega\frac{\underline{a}_{i_j}e^j}{d_{i_j}}\underline{a}_{i_j}^H - \frac{\omega\hat b_{i_j}}{d_{i_j}}\underline{a}_{i_j}^H,$$

where $i_j = j \pmod m + 1$, $j \ge 0$. Thus, when combining the first m steps, we get $\bar e^1 = Qe^0 + R\hat b$ with certain matrices Q, R depending on A, D, and ω. Under the assumptions of Theorem 1, Q is contractive on $\mathrm{Ran}(A^H)$ and obviously $\mathrm{Ran}(R) \subset \mathrm{Ran}(A^H)$. Thus,

$$\bar e^\ell = Q^\ell e^0 + (I + Q + \ldots + Q^{\ell-1})R\hat b,$$

and the iterates x^j converge to $x_{LS} - (I-Q)^{-1}R\hat b$, still under the assumption that $x^0 \in V_a = \mathrm{Ran}(A^H)$. The error estimate after ℓ cyclic sweeps is

$$\|x_{LS} - \bar x^\ell\|_2 \le \|Q\|_2^\ell\|x_{LS} - x^0\|_2 + \frac{\|R\hat b\|_2}{1-\|Q\|_2}, \qquad \bar x^\ell = x^{\ell m},$$

where, according to Theorem 4 the spectral norm of $Q : V_a \to V_a$ is bounded by

$$\|Q\|_2^2 \le 1 - \frac{C}{(\ln(r)+1)\bar\kappa(A^HD^{-1}A)}.$$

Thus the iteration converges to a point that sits in a ball of radius proportional to $\|\hat{b}\|_2$ around x_{LS} (see [10, Theorem 4.32] for a formula expression of the limit).

A similar approach applies to the random Kaczmarz iteration resulting from Theorem 2. Indeed, the error recursion for the update (38) can be written as

$$e^{j+1} = \left(e^j - \frac{\underline{a}_{i_j} e^j}{\|\underline{a}_{i_j}\|_2^2} \underline{a}_{i_j}^H\right) + \frac{(1-\omega)\underline{a}_{i_j} e^j - \omega \hat{b}_{i_j}}{\|\underline{a}_{i_j}\|_2^2} \underline{a}_{i_j}^H,$$

with the two terms orthogonal to each other. Thus, a quick computation shows

$$\|e^{j+1}\|_2^2 = (\|e^j\|_2^2 - \frac{|\underline{a}_{i_j} e^j|^2}{\|\underline{a}_{i_j}\|_2^2}) + \frac{|(1-\omega)\underline{a}_{i_j} e^j - \omega \hat{b}_{i_j}|^2}{\|\underline{a}_{i_j}\|_2^2}.$$

For $\omega = 1$ (this is the case covered in [1]), we continue with computing the conditional expectation of $\|e^{j+1}\|_2^2$ with respect to given e^j, recall that the probability distribution underlying the choice of i_j is given by (37):

$$\begin{aligned} E(\|e^{j+1}\|_2^2 \,|\, e^j) &\le \|e^j\|_2^2 - \sum_{i=1}^m \frac{|\underline{a}_i e^j|^2}{d_i \|D^{-1/2}A\|_F^2} + \sum_{i=1}^m \frac{|\hat{b}_i|^2}{d_i \|D^{-1/2}A\|_F^2} \\ &\le \left(1 - \frac{\|D^{-1/2}Ae^j\|_2^2}{\|e^j\|_2^2 \|D^{-1/2}A\|_F^2}\right) \|e^j\|_2^2 + \frac{\|D^{-1/2}\hat{b}\|_2^2}{\|D^{-1/2}A\|_F^2} \\ &\le \left(1 - \frac{\sigma_{\min}^2(D^{-1/2}A)}{\|D^{-1/2}A\|_F^2}\right) \|e^j\|_2^2 + \frac{\|D^{-1/2}\hat{b}\|_2^2}{\|D^{-1/2}A\|_F^2}. \end{aligned}$$

Taking the expectation with respect to e^j, we get

$$E(\|e^{j+1}\|_2^2) \le \left(1 - \frac{\sigma_{\min}^2(D^{-1/2}A)}{\|D^{-1/2}A\|_F^2}\right) E(\|e^j\|_2^2) + \frac{\|D^{-1/2}\hat{b}\|_2^2}{\|D^{-1/2}A\|_F^2}, \qquad j \ge 0,$$

and iterating this inequality results in

$$E(\|x_{LS} - x^j\|_2^2) \le \left(1 - \frac{\sigma_{\min}^2(D^{-1/2}A)}{\|D^{-1/2}A\|_F^2}\right)^j \|x_{LS} - x^0\|_2^2 + \frac{\|D^{-1/2}\hat{b}\|_2^2}{\sigma_{\min}^2(D^{-1/2}A)}, \qquad j \ge 1.$$

For $\omega \ne 1$, one can first use the elementary inequality

$$(a+b)^2 \le (1+t)a^2 + (1+\frac{1}{t})b^2, \qquad a, b, t > 0,$$

with

$$t = (1-C_2)\frac{\omega(2-\omega)}{(1-\omega)^2}, \quad a = \frac{(1-\omega)|\underline{a}_{i_j} e^j|}{\|\underline{a}_{i_j}\|_2}, \quad b = \frac{\omega|\hat{b}_{i_j}|}{\|\underline{a}_{i_j}\|_2}, \quad 0 < C_2 < 1,$$

and then repeat the computation of expectations. This leads to a slightly worse recursive estimate

$$E(\|e^{j+1}\|_2^2) \le \left(1 - \frac{C_2\omega(2-\omega)\sigma_{\min}^2(D^{-1/2}A)}{\|D^{-1/2}A\|_F^2}\right) E(\|e^j\|_2^2) + \frac{(1-C_2)\omega(2-\omega)\omega\|D^{-1/2}\hat{b}\|_2^2}{(1-C_2)(2-\omega)\|D^{-1/2}A\|_F^2},$$

$j \ge 0$, but a similar conclusion: For inconsistent systems, the random Kacmarz iteration stabilizes in expectation (at almost the same linear convergence rate as in the consistent case) into a ball around x_{LS} with radius proportional to $\|D^{-1/2}\hat{b}\|_2$ measuring the inconsistency of the right-hand side b. Similar arguments can be provided for the block-random Kaczmarz iteration covered by Theorem 3 and for the block-Kaczmarz iterations discussed in the next subsection.

3.2. Kaczmarz methods: block updates and least-squares solvers

Block-iterative methods for general linear systems [27] often lead to better cpu-time efficiency in implementations, even though this cannot always be substantiated theoretically. In the language of ADM, this means to go away from one-dimensional search directions given by the columns of A^H (and projections onto hyperplanes), and replace them by more general search directions or subspace search. For reasons explained in the previous subsection, we can w.l.o.g. assume that $Ax = b$ is consistent, i.e., $b \in \mathrm{Ran}(A)$, and that $x^0 \in \mathrm{Ran}(A^H)$.

Block-Kaczmarz iterations based on a row partitioning $\mathcal{T}$ as introduced in Example 3 have been proposed in slightly more general form in [27], the update formula reads

$$x^{j+1} = x^j + \omega_j A_{\tau_{k_j}}^{\dagger}(b_{\tau_{k_j}} - A_{\tau_{k_j}} x^j). \tag{43}$$

The more recent papers [4] and [5, Algorithm 1] deal with randomized versions, under the assumption that the row partition $\mathcal{T}$ leads to invertible and well-conditioned matrices $A_{\tau_k} A_{\tau_k}^H$. It is easy to check that the iteration (15) based on the splitting from Example 3 leads to exactly the update formula (43), which allows us to deduce convergence results for both cyclic and randomized block-Kaczmarz iterations from the theorems in Section 2. In particular, for the cyclic ordering $k_j = j \pmod K + 1$, $j \ge 0$, we obtain from Theorem 1 that

$$\|x_{LS} - \bar{x}^\ell\|_2^2 \le \left(1 - \frac{1}{(\lfloor \log_2(2K)\rfloor \kappa_{\mathcal{T}}}\right)^\ell \|x_{LS} - x^0\|_2^2,$$

where $\bar{x}^\ell = x^{\ell K}$, $\ell \ge 1$, if ω is chosen properly. Here, $\kappa_{\mathcal{T}}$ is the condition number of the splitting in Example 3. Similarly, since for this splitting obviously $\gamma_k = 1$, $k = 1, \ldots, K$, Theorem 2 implies

$$E(\|x_{LS} - x^j\|_2^2) \le \left(1 - \frac{\lambda_{\min,\mathcal{T}}}{K}\right)^j \|x_{LS} - x^0\|_2^2, \qquad j \ge 1,$$

for the randomized block-Kaczmarz iteration with underlying uniform probability distribution and relaxation parameter $\omega = 1$. The estimate remains valid if ω_j is computed by the steepest descent formula. As was mentioned in Section 2.1, the stability constants $\lambda_{\max\,/\,\min,\mathcal{T}}$ are hard to assess for general row partitions $\mathcal{T}$. For $\mathcal{T}$ satisfying (6), the estimation of the condition number $\kappa_{\mathcal{T}}$ leading to (7) reveals that $\lambda_{\min,\mathcal{T}} \ge \sigma_{\min}^2(A)/\beta$, and we arrive at exactly the result of [4, Theorem 1.2] for the consistent case (the inconsistent case $b \notin \mathrm{Ran}(A)$ can be handled as described in Section 3.1). We refer to [4] for a discussion of the state of the art of finding partitions $\mathcal{T}$ with properties close to (6).

Our approach can be used to design and analyze other block iterations. For instance, if we change in Example 3 the auxiliary spaces to $V_{a_k} = \mathbb{C}^{|\tau_k|}$, and the auxiliary scalar products to $a_k(y_{\tau_k}, y_{\tau_k}) = \sum_{i\in\tau_k} d_i |y_{\tau_k,i}|^2$ but keep all other components of the space splitting as they are, then it is not hard to see that the stability constants and condition numbers for this modified space splitting coincide with those of the splitting from Example 2, in particular, $\kappa = \bar{\kappa}(A^H D^{-1} A)$. Since $T_k x = (D^{-1}A)_{\tau_k} x$, we get the update formula

$$x^{j+1} = x^j + \omega A_{\tau_{k_j}}^H D_{\tau_{k_j}}^{-1} (b_{\tau_{k_j}} - A_{\tau_{k_j}} x^j), \qquad k_j = j \pmod K + 1, \tag{44}$$

for cyclic orderings, and an error estimate of

$$\|x_{LS} - \bar{x}^\ell\|_2^2 \le \left(1 - \frac{1}{(\lfloor \log_2(2K)\rfloor + 1)\bar{\kappa}(A^H D^{-1} A)}\right)^\ell \|x_{LS} - x^0\|_2^2,$$

where again $\bar{x}^\ell = x^{\ell K}$, $\ell \ge 1$, and ω is chosen properly.

For random orderings, the update reads

$$x^{j+1} = x^j + \frac{\omega}{\gamma_{k_j}} A_{\tau_{k_j}}^H D_{\tau_{k_j}}^{-1} (b_{\tau_{k_j}} - A_{\tau_{k_j}} x^j), \qquad j \ge 0, \tag{45}$$

and $k = k_j$ is picked from the index range $\{1, \dots, K\}$ according to the probability distribution $p_k = \gamma_k/(\gamma_1 + \dots + \gamma_K)$, where $\gamma_k = \|(D^{-1/2}A)_{\tau_k}\|_2^2$, $k = 1, \dots, K$. The difference of the update in (45) with the similar update formula (41) obtained from Theorem 3 is that there we choose uniformly randomly an arbitrary subset $I_j \subset \{1, \dots, m\}$, whereas now we only pick randomly an index subset among the subsets τ_k from a fixed partition $\mathcal{T}$. However, the estimates for the expected square error are similar. Indeed, the application of Theorem 2 to the iteration (45) yields

$$E(\|x_{LS} - x^j\|_2^2) \le \left(1 - \frac{\omega(2-\omega)\sigma_{\min}^2(D^{-1/2}A)}{\sum_{k=1}^K \|(D^{-1/2}A)_{\tau_k}\|_2^2}\right)^j \|x_{LS} - x^0\|_2^2, \qquad j \ge 1,$$

if $\omega \in (0,2)$. Again, this is not too explicit as the constants γ_k depend on the partition $\mathcal{T}$, it is, however, worth mentioning that

$$\sigma_{\max}^2(D^{-1/2}A) \le \sum_{k=1}^{K} \|(D^{-1/2}A)_{\tau_k}\|_2^2 \le \|D^{-1/2}A\|_F^2.$$

We conclude this subsection with a few remarks on obtaining the least-squares solution $x_{LS} = A^\dagger b$ for inconsistent linear systems $Ax = b$, where the Kaczmarz-type algorithms discussed so far converge only to a point in a neighborhood of x_{LS} of radius proportional to the distance of b to $\mathrm{Ran}(A)$. One work-around are the algorithms proposed in [3,5] which are based on applying Kaczmarz-type iterations in an alternating fashion to the block-triangular system

$$Ax = b - y, \qquad A^H y = 0,$$

with starting vectors $x^0 = 0$, $y^0 = b$. According to the above theory, the iterates y^j converge to $\hat{b}$ (the projection of b onto $\mathrm{Ker}(A^H)$), and since $b - \hat{b} \in \mathrm{Ran}(A)$ as well as $x^0 \in \mathrm{Ran}(A^H)$, the iterates x^j must converge to x_{LS}. The following extended randomized Kaczmarz method was proposed and analyzed in [3]: Given x^0, y^0, for $j = 0, 1, \ldots$, choose a column $\bar{a}_{k_j}$ of A, where $k_j \in \{1, \ldots, n\}$ are i.i.d. random variables with discrete probability distribution $q_k = \|\bar{a}_k\|_2^2/\|A\|_F^2$, $k = 1, \ldots, n$, and set

$$y^{j+1} = y^j - \omega \frac{\bar{a}_{k_j}^H y^j}{\|\bar{a}_{k_j}\|_2^2} \bar{a}_{k_j}. \tag{46}$$

Next, choose a row $\underline{a}_{i_j}$ of A, where $i_j \in \{1, \ldots, m\}$ is an i.i.d. random variable with discrete probability distribution $p_i = \|\underline{a}_i\|_2^2/\|A\|_F^2$, $i = 1, \ldots, m$, and update

$$x^{j+1} = x^j + \omega \frac{b_{i_j} - \underline{a}_{i_j} x^j - y_{i_j}^j}{\|\underline{a}_{i_j}\|_2^2} \underline{a}_{i_j}^H. \tag{47}$$

A block version of this algorithm has been considered in [5], where it is assumed that both A and A^H admit, after respective column and row scaling, pavings with constants α, β in the corresponding assumptions (6) such that $\beta/\alpha \approx 1$, thus fitting the discussion of the block-Kaczmarz iterations in the previous subsection. Cyclic versions are possible as well: One would perform a full sweep for $A^H y = 0$ to update from $\bar{y}^\ell$ to $\bar{y}^{\ell+1}$, followed by a full sweep for $Ax = b - y^\ell$ (or, equally well for $Ax = b - y^{\ell+1}$) to update from $\bar{x}^\ell$ to $\bar{x}^{\ell+1}$. Finally, instead of alternating between y and x updates, one can also leave the decision to the randomization process: In the j-th step, with equal probability, choose first whether to update x^j or y^j. For an x-update, compute x^{j+1} according to the instructions in (47) and set $y^{j+1} = y^j$ while for a y-update y^{j+1} is obtained by (46) and $x^{j+1} = x^j$. The analysis of all these versions for finding x_{LS} can be done using

the framework of Schwarz iterative methods using appropriate splittings, with the additional advantage that this theory also provides convergence estimates if cyclic orderings are preferred.

4. Numerical tests

In this section we illustrate some of the main results of this paper, in particular, the convergence bounds for Kaczmarz iterations with cyclic ordering in comparison with random orderings, by numerical experiments. Test matrices from three families of matrices are considered below. The first one (referred to as Toeplitz matrices) is taken from [28], and consists of finite $m \times n$ sections $A = ((A_{j-k}))_{j=1,\dots,m,\ k=1,\dots,n}$ of a Hermitian positive-definite Toeplitz operator on $\ell^2(\mathbb{Z})$ given by the sequence

$$A_0 = 1, \qquad A_{2k} = 0, \quad A_{\pm(2k-1)} = \frac{c_0(-1)^{k-1}}{2k-1}, \qquad k = 1, 2, \dots, \quad 0 < c_0 < \frac{2}{\pi}.$$

We have chosen $c_0 = 0.2$ in our tests. All matrices A chosen from this family have full rank $r = \min(m, n)$, and are well-conditioned with almost constant condition numbers $\bar{\kappa}(A^H A) \approx 3.671$.

The second family (referred to as Fourier matrices) originates from the classical problem of reconstructing 1-periodic band-limited functions from samples at non-uniformly spaced points, and was already used in, e.g., [6]. Let A be defined by its entries as

$$A_{jk} = \sqrt{w_j} e^{2\pi i k t_j}, \qquad w_j = \frac{t_{j+1} - t_{j-1}}{2},$$

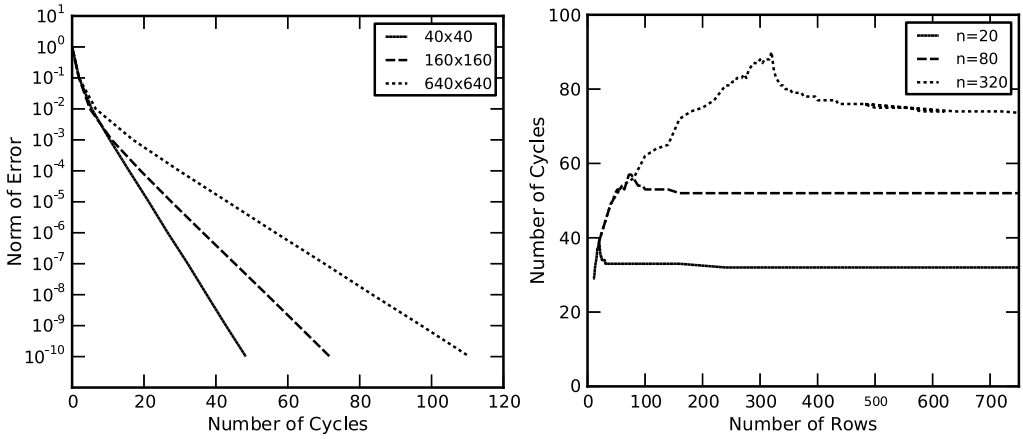
where $\{t_j\}_{j=1,\dots,m}$ is an increasing sequence of non-uniformly spaced sampling points in the periodic unit interval drawn from a uniform distribution, and $k = -K, \dots, K$ (i.e., $n = 2K + 1$). As was justified in [6], the introduction of the above weight factors w_j, and a sufficiently large oversampling rate $m/n >> 1$ guarantee that condition numbers fall in a reasonable range. In our experiments, we chose $K = 50$ and $m = 500$, and generated A of size 500×101, with full column rank $r = n = 101$, and with $\bar{\kappa}(A^H A) \approx 312.5$.

Finally, a third family (referred to as Tomography matrices) was generated using the Matlab Regularization Toolbox by P.C. Hansen, described in [29] and available at `http://www.imm.dtu.dk/~pcha/Regutools/`. Its routine `tomo` allows for the generation of certain 2D tomography problems $Ax = b$ of size $m = fN^2$ by $n = N^2$, where N and the oversampling rate f are user-supplied constants. Each row in A corresponds to the absorption characteristics along a randomly placed line through a $N \times N$ box grid. In our tests, we chose $N = 20$ and various values for f (below, results are reported only for $f = 3$). Again, if the oversampling rate f is significantly larger than 1, the condition number $\bar{\kappa}(A^H A)$ tends to become reasonably small while for $f \approx 1$ it may become very large, mostly due to a few very small non-zero singular values of A.

In our experiments with the first two families, we solved the homogeneous problem $Ax = 0$ with a starting unit vector x^0 in $\mathrm{Ran}(A^H)$, hence the iteration converges to $x_{LS} = 0$. The vector x^0 is randomly chosen but kept fixed in all experiments, thus the results are comparable. For the third family, we solved the consistent problem $Ax = b$ with b and $x = x_{LS}$ supplied by the routine `tomo`. Our primary index of performance measurement is the number of sweeps or cycles needed to reach a given precision level (measured by relative error). The cycle count is defined as number of single row updates divided by m for the standard Kaczmarz iteration, and analogously as number of block updates multiplied by k/m for block-Kaczmarz iterations with constant block size k. Finally, unless otherwise specified, the shown graphs depict always the average of 5 independent experiments whenever the iteration involves random row/block selection or random row permutations.

Our tests concentrate on the following questions related to the theoretical material of this paper:

1) Our convergence rate estimates for multiplicative Schwarz iterations, and in particular Kaczmarz iterations, with cyclic orderings deteriorate logarithmically in the number of subproblems resp. the rank of A (Theorems 1 and 4). That such a deterioration cannot be excluded, is shown by tests with the first family of matrices. For the other matrix families related to more natural recovery problems from sampled information, such an effect is not visible.
2) The actual convergence of the cyclic iteration depends on the ordering of equations in $Ax = b$ although the convergence bounds in Theorems 1 and 4 do not reflect this: Premultiplying the system by any $m \times m$ permutation matrix P_π neither changes the rank r nor the condition number $\bar{\kappa}(A^H A)$. We have tested the behavior of Kaczmarz iterations with cyclic ordering after a random row permutation was applied (random row shuffling followed by Kaczmarz iteration with cyclic ordering). Such a simple preprocessing leads to results at least as good as achieved by iterations with random orderings.
3) We also implemented randomized block-Kaczmarz iterations, where in each step we chose, uniformly at random, row subsets of fixed but small size $k > 1$, and performed updates of the form (44) (called for short Jacobi updates) or (43) (called for short least-squares updates). In the reported tests, due to the lack of good information on the scaling constants γ_i, we have defined relaxation parameters by the steepest decent formula (32). The main observation is that, at least for the considered families of problems and relatively small block size k, the different randomized block-Kaczmarz iterations possess similar convergence behavior but do not outperform standard Kaczmarz iterations in relation to the overall number of rows touched during the iteration process.



(a) Error decay for growing size

(b) Cycle counts for fixed n to reach 10^{-10}

Fig. 1. Necessity of the logarithm factor.

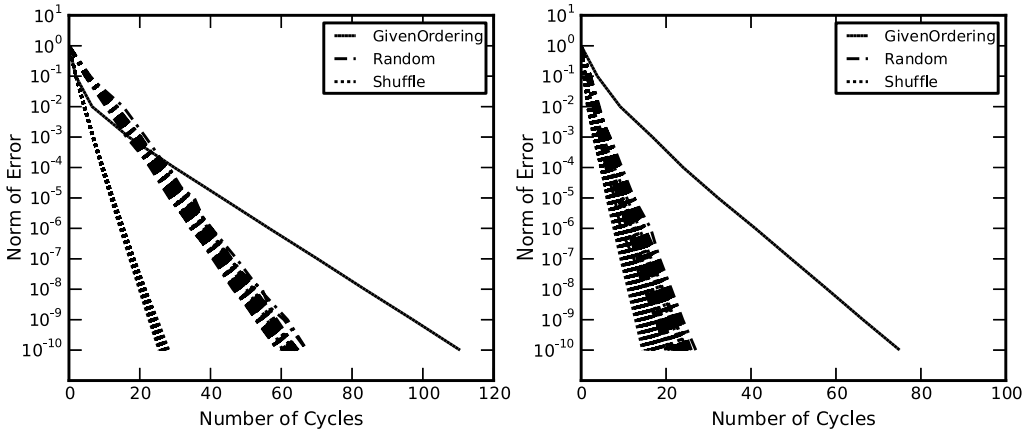
4.1. The logarithmic factor

In the first experiment (Fig. 1 (a)), we applied the Kaczmarz iteration with cyclic ordering to linear systems with square matrices A of sizes 40×40, 160×160, 640×640 from the first family (Toeplitz matrices). As the matrix size quadruples, a linear decay of the error reduction rate can be clearly observed. Since the condition numbers for the matrices from this family remain almost perfectly constant, this demonstrates the necessity of the logarithmic factor in the error estimate of Theorem 2.

The second experiment (Fig. 1 (b)) aims at illustrating the relevance of the $\ln(r)$ factor in Theorem 4. We applied the cyclic Kaczmarz method to A from the first family (Toeplitz matrices) with variable row dimension $20 \leq m < 800$ but fixed column dimension $n = 20, 80, 320$ respectively, and recorded the cycles needed for the error norm to drop below 10^{-10}. It can be clearly seen that the error reduction rate is dependent on the rank $r = \min(m, n)$.

4.2. Row shuffling

As mentioned before, the error decay bound for cyclic orderings stated in Theorem 4 is invariant under row shuffling. However, the actual convergence rates may well change, as the lower triangular matrix L_{AA^H} crucially enters the estimates and depends on row permutation. We did experiments on both the first (Toeplitz matrix of size 640×640) and second (Fourier matrix of size 500×101) family of matrices, in each of the 50 recorded experiments we used cyclic iteration but with different fixed row ordering. As a comparison we also included experiments using the random Kaczmarz iteration described in Theorem 2. For both families of matrices, one-time row shuffling followed by cyclic Kaczmarz iteration outperforms the cyclic Kaczmarz iteration in the initially given order, and is even better than the random Kaczmarz iteration (see Fig. 2). This



(a) Row shuffling on Toeplitz Matrix (b) Row shuffling on Fourier Matrix

Fig. 2. Improvement by row shuffling.

experiment suggests that a simple preprocessing step of a one-time row shuffling before cyclic iteration, especially in applications where we are not sure if the given ordering is optimal, may lead to performance as good as for iterations with more sophisticated or costly randomization strategies. Since one cycle of the Kaczmarz iteration after random row shuffling is equivalent to a cycle of a Kaczmarz iteration where indices are chosen randomly but without repetition, this is in line with the often observed behavior of randomized iteration schemes with and without repetition, see [30] for a discussion of this aspect.

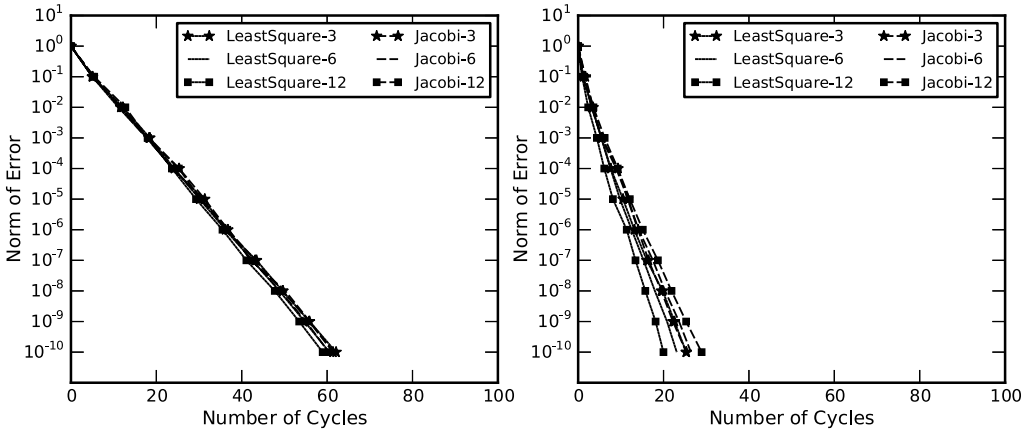
4.3. Block iterations

In Fig. 3 (a) and (b), we show error decay graphs for block-Kaczmarz iterations for the 640×640 Toeplitz matrix, and another matrix from the second family (Fourier matrix) respectively. We implemented (44) and (43), both with fixed block-size $k_j = k = 3, 6, 12$, and randomly chosen τ_{k_j}. The first version (Jacobi update) is based on Theorem 3, and uses $B_{\tau_{k_j}} = A^H_{\tau_{k_j}}$, the second (least-squares update) uses $B_{\tau_{k_j}} = A^\dagger_{\tau_{k_j}}$ as suggested in [4,5]. In both cases, ω_j is chosen according to the steepest descent rule, i.e., $\omega_j = 1$ for the least-squares update, and

$$\omega_j = \frac{\|r^j\|_2^2}{\|A^H_{\tau_{k_j}} r^j\|_2^2}, \qquad r^j := b_{\tau_{k_j}} - A_{\tau_{k_j}} x^j,$$

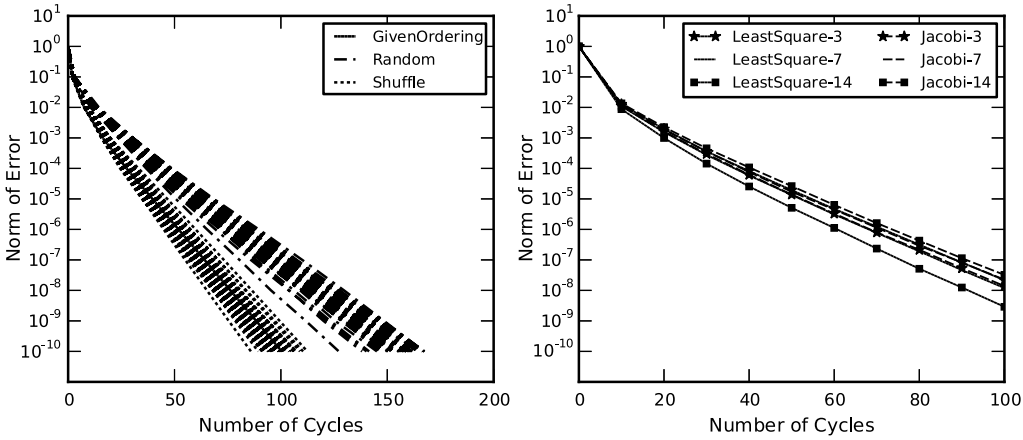
for the Jacobi update.

The results for the Jacobi update case are in full agreement with the bounds given in Theorem 3, for small block-sizes $k = k_j$ the cycle count for reaching a certain error reduction only slightly increases with k. Moreover, for this range of k, the two methods are of comparable performance and cost. Note that, to our knowledge, there is no theoretical convergence rate bound available for the implemented version with least-squares



(a) on Toeplitz Matrix (b) on Fourier Matrix

Fig. 3. Block iteration.



(a) Single row iteration (b) Block iteration

Fig. 4. Iterations on Tomography matrix.

updates. One can only speculate that this method becomes more competitive as k_j is chosen larger, at the expense of increased computational cost compared to the simpler Jacobi updates.

4.4. Tomography matrices

For the tests illustrated by Fig. 4, we generated a matrix A of size 1200×400 and with condition number $\bar{\kappa}(A^H A) \approx 625$ from the tomography family using the routine `tomo` with parameters $N = 20$ and $f = 3$. As shown in Fig. 4 (a), in contrast to the previously reported findings, the cyclic Kaczmarz iteration (without any row shuffling) behaves as good as shuffled versions. The reason might be that the rows of matrices created by `tomo` are already shuffled, as they result from taking intensity measurements

along a set of randomly chosen lines crossing a two-dimensional grid. The tests with block-Kaczmarz iterations reported in Fig. 4 (b) are in line with the observations from the previous subsection.

References

[1] D. Needell, Randomized Kaczmarz solver for noisy linear systems, BIT 50 (2) (2010) 395–403.
[2] Y.C. Eldar, D. Needell, Acceleration of randomized Kaczmarz method via the Johnson–Lindenstrauss lemma, Numer. Algorithms 58 (2) (2011) 163–177.
[3] A. Zouzias, N.M. Freris, Randomized extended Kaczmarz for solving least squares, SIAM J. Matrix Anal. Appl. 34 (2) (2013) 773–793.
[4] D. Needell, J.A. Tropp, Paved with good intentions: analysis of a randomized block Kaczmarz method, Linear Algebra Appl. 441 (2014) 199–221.
[5] D. Needell, R. Zhao, A. Zouzias, Randomized block Kaczmarz method with projection for solving least squares, arXiv preprint, arXiv:1403.4192.
[6] T. Strohmer, R. Vershynin, A randomized Kaczmarz algorithm with exponential convergence, J. Fourier Anal. Appl. 15 (2) (2009) 262–278.
[7] M. Griebel, P. Oswald, On the abstract theory of additive and multiplicative Schwarz algorithms, Numer. Math. 70 (2) (1995) 163–180.
[8] M. Griebel, P. Oswald, Greedy and randomized versions of the multiplicative Schwarz method, Linear Algebra Appl. 437 (7) (2012) 1596–1610.
[9] F. Deutsch, Best Approximation in Inner Product Spaces, CMS Books Math., vol. 7, Springer, 2001.
[10] A. Galántai, Projectors and Projection Methods, Adv. Math., vol. 6, Springer, 2004.
[11] H.H. Bauschke, J.M. Borwein, On projection algorithms for solving convex feasibility problems, SIAM Rev. 38 (3) (1996) 367–426.
[12] G.T. Herman, Fundamentals of Computerized Tomography: Image Reconstruction from Projections, Springer, 2009.
[13] S. Kaczmarz, Angenäherte Auflösung von Systemen linearer Gleichungen, Bull. Int. Acad. Pol. Sci. Lett. 35 (1937) 355–357.
[14] J. Xu, Iterative methods by space decomposition and subspace correction, SIAM Rev. 34 (4) (1992) 581–613.
[15] H. Yserentant, Old and new convergence proofs for multigrid methods, Acta Numer. 2 (1993) 285–326.
[16] P. Oswald, Multilevel Finite Element Approximation: Theory and Applications, Teubner, Stuttgart, 1994.
[17] D. Leventhal, A.S. Lewis, Randomized methods for linear constraints: convergence rates and conditioning, Math. Oper. Res. 35 (3) (2010) 641–654.
[18] P.G. Casazza, M. Fickus, J.C. Tremain, E. Weber, The Kadison–Singer problem in mathematics and engineering: a detailed account, Contemp. Math. 414 (2006) 299.
[19] A. Marcus, D.A. Spielman, N. Srivastava, Interlacing families II: mixed characteristic polynomials and the Kadison–Singer problem, arXiv preprint, arXiv:1306.3969.
[20] J. Xu, L. Zikatanov, The method of alternating projections and the method of subspace corrections in Hilbert space, J. Amer. Math. Soc. 15 (3) (2002) 573–597.
[21] R. Meany, A matrix inequality, SIAM J. Numer. Anal. 6 (1) (1969) 104–107.
[22] Y. Nesterov, Efficiency of coordinate descent methods on huge-scale optimization problems, SIAM J. Optim. 22 (2) (2012) 341–362.
[23] P. Richtárik, M. Takáč, Iteration complexity of randomized block-coordinate descent methods for minimizing a composite function, Math. Program. 144 (1–2) (2014) 1–38.
[24] Y. Censor, T. Elfving, G.T. Herman, T. Nikazad, On diagonally relaxed orthogonal projection methods, SIAM J. Sci. Comput. 30 (1) (2008) 473–504.
[25] M. Griebel, A. Hullmann, P. Oswald, Optimal scaling parameters for sparse grid discretizations, Numer. Linear Algebra Appl. 22 (1) (2015) 76–100.
[26] I. Gohberg, M.G. Krein, Theory and Applications of Volterra Operators in Hilbert Space, Transl. Math. Monogr., vol. 24, Amer. Math. Soc., 1970.
[27] T. Elfving, Block-iterative methods for consistent and inconsistent linear equations, Numer. Math. 35 (1) (1980) 1–12.
[28] P. Oswald, On the convergence rate of SOR: a worst case estimate, Computing 52 (3) (1994) 245–255.

[29] P.C. Hansen, Regularization tools version 4.0 for Matlab 7.3, Numer. Algorithms 46 (2) (2007) 189–194.
[30] B. Recht, C. Ré, Toward a noncommutative arithmetic–geometric mean inequality: conjectures, case-studies, and consequences, in: COLT 2012, in: J. Mach. Learn. Res. Workshop Conf. Proc., vol. 23, 2012, pp. 11.1–11.24.